Relationship Between Log-Ratio and Difference-Over-Sum Characterizations of Beam Position Monitor Sensitivity

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There are numerous ways to characterize the sensitivity of Beam Position Monitors to achieve the ultimate goal of determining the beam position from the amplitude of signals from opposing BPM electrodes (or combinations of electrodes). Two common models describe the position as a function of the electrode signals, A and B, using:

1) the logarithmic ratio of the signals --- usually 20 log (A/B)
2) the normalized difference of the signals --- (A-B)/(A+B)

Where the relationship between the position and the log ratio of the signals is linear over the region of interest, the BPM may be characterized by a constant sensitivity, usually in mm/dB. Note that the dB is unitless, so the actual unit of the constant is just mm. Position is computed by multiplying that constant by the dB ratio of the signal amplitudes.

Where there is a linear relationship between position and the normalized signal difference, the BPM is characterized by an "effective radius" and position is computed by multiplying the effective radius by the difference-over-sum value.

Non-linear relationships between position and the signal representations are of course also possible and, in fact, common for many BPM electrode designs.

The Mathematical Relationship

The relationship between the "logarithmic or db ratio" description and the "normalized difference or difference over sum" description is shown below.

Begin with the expansion:

$$\ln\left(\frac{1+z}{1-z}\right) = 2\cdot z + \frac{2}{3}\cdot z^3 + \frac{2}{5}\cdot z^5 + O\left(z^6\right)$$

Substitute $z = \dfrac{A-B}{A+B}$ to get:

$$\ln\left(\frac{A}{B}\right) = 2\cdot\frac{A-B}{A+B} + \frac{2}{3}\cdot\frac{(A-B)^3}{(A+B)^3} + \frac{2}{5}\cdot\frac{(A-B)^5}{(A+B)^5} + O\left[\frac{(A-B)^6}{(A+B)^6}\right]$$

Using $\log(x) = \dfrac{\ln(x)}{\ln(10)}$, find:

$$20\cdot\log\left(\frac{A}{B}\right) = \frac{40}{\ln(10)}\cdot\left[\frac{A-B}{A+B} + \frac{1}{3}\cdot\frac{(A-B)^3}{(A+B)^3} + \frac{1}{5}\cdot\frac{(A-B)^5}{(A+B)^5} + O\left[\frac{(A-B)^6}{(A+B)^6}\right]\right]$$

This then is the relationship between the dB signal ratio and the difference-over-sum quantity.

An Example BPM Application

Assume a certain BPM is characterized by a linear relation between position and the dB signal ratio. (Note: this is an example, it does not necessarily represent any particular real BPM.) Let the characteristic sensitivity constant be $K_{mmperdb}$ mm/dB. The position in millimeters is computed as:

$$x = K_{mmperdb}\cdot 20\cdot\log\left(\frac{A}{B}\right)$$

or

$$x = K_{mmperdb}\cdot\frac{40}{\ln(10)}\cdot\left[\frac{A-B}{A+B} + \frac{1}{3}\cdot\frac{(A-B)^3}{(A+B)^3} + \frac{1}{5}\cdot\frac{(A-B)^5}{(A+B)^5} + O\left[\frac{(A-B)^6}{(A+B)^6}\right]\right]$$

In the difference-over-sum expansion the corresponding "effective radius" is:

$$R_{eff} = K_{mmperdb}\cdot\frac{40}{\ln(10)} = 17.37\cdot K_{mmperdb}$$

If some BPM signal processing electronics produces results of magnitudes A and B, rather than directly the log ratio of A/B, it might be tempting to compute position using only the linear term of the difference-over-sum expansion. What errors would result from this approximation?

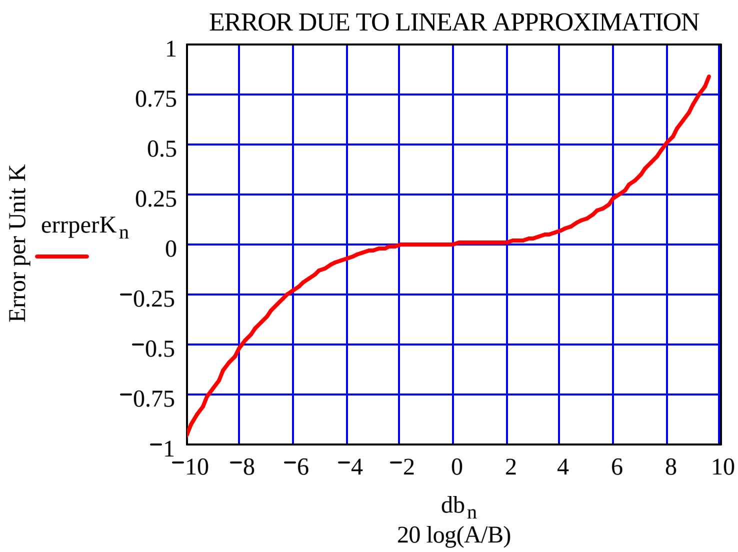
The results for A/B in the range of 1/3 to 3 are computed and plotted below. Since only the ratio of A and B is of consequence, any value may be chosen for either A or B or A+B. Here the sum is held constant, A+B=1.

$n := 0..100 \qquad rat_n := 3\cdot 10^{\frac{n}{100}-1} \qquad db_n := 20\cdot\log(rat_n)$

$B_n := \dfrac{1}{1+rat_n} \qquad A_n := B_n\cdot rat_n \qquad dos_n := \dfrac{A_n - B_n}{A_n + B_n}$

Considering terms only to fifth order, the error per unit K is:

$$errperK_n := \frac{40}{\ln(10)} \cdot \left[\frac{1}{3} \cdot \frac{(A_n - B_n)^3}{(A_n + B_n)^3} + \frac{1}{5} \cdot \frac{(A_n - B_n)^5}{(A_n + B_n)^5} \right]$$



The graph shows the resulting error if a BPM, linear in dB/mm, is approximated as linear in difference-over-sum. As might be expected, errors are small for positions near center, but grow rapidly for signal ratios above 3 dB. For instance, an error of 1 mm results for a 4 mm/dB BPM at a signal ratio is 6 dB if position is computed from difference-over-sum using only the linear term of the expansion. Since the numerical signs of the higher order terms are the same as that of the linear term, neglecting non-linear terms results in a reported position that is "too small" in magnitude.

Remember that this is only an example and does not imply correctness of the original assumption of a linear relation between position and log signal ratio for any particular BPM.